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Velocity and two-directional focusing of charged particles in crossed electric and magnetic fields. I

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1. Introduction

In all systems intended for focusing of charged particles and giving energy or momentum dispersion, one has always to compromise between the demand of high resolving power and that of a high transmitting power. When one goes beyond the most elementary focusing systems, e. g. those achieved by homogeneous magnetic fields, there exists an infinite number of possibilities to combine magnetic and electrostatic fields in order to increase one of the two abovementioned factors without any decrease of the other. Introductory theories treating such more general systems, have been developed among others by COTTE (1938) and WENDT (1942). It seems adequate, however, to select for a more close study some special groups, which are characterized both by simplicity and practical advantages. In the present paper the treatment is limited to field combinations possessing cylindrical symmetry. It is also aimed to be an introduction to a theory of a mass spectrometer for mass defect work which is being built at this Institute. An essential feature of this apparatus is the realization of almost ideal focusing conditions and the avoidance of fringing fields in the ordinary sense.

In the mathematical treatment of the aberrations of such focusing systems, one has to choose between the eiconal method and the path equation method. Since in either case the cylinder symmetrical fields are unique to give explicit formulae, the author preferred the latter method. In more general cases where the aberration entities are obtained as integrals containing functions, which are defined only by differential equations, the eiconal method has evidently some advantages especially in the systematization of the aberrations. A thourough discussion of the latter ones is made by MARSCHALL (1944) in a paper, which was brought to the author's notice by Professor S. FLÜGGE at the Birmingham conference in 1948. I wish to thank Professor FLÜGGE for his communication and Dr. MARSCHALL for his kindness in sending me a copy of his paper, which has not been available here. Though aberration formulae for more general field forms are given, it seems as if just the eiconal method, which is used in that paper, has prevented the author to find the most interesting special focusing systems exhibited by fields of cylindrical symmetry.

2. The electric field

The electrical as well as the magnetic field is assumed to have cylindrical symmetry with respect to the z-axis and mirror symmetry with respect to the xy-plane. If the electrical field strength is given within an annular region of the latter plane and is expressed by a series expansion in the vicinity of the circle $r = r_0$

$$\mathfrak{E}_r(r, 0) = E(r) = E_0 (1 + \beta_1 \varrho + \beta_2 \varrho^2 + \cdots) \qquad (2:1)$$

where

$$\varrho = (r - r_0)/r_0$$

one can compute the potential and, consequently, the field components $\mathfrak{E}_r(r, z)$ and $\mathfrak{E}_z(r, z)$ within an annular spatial region in the vicinity of the above-mentioned circle. Putting

$$\Phi(r, z) = \sum_{\nu=0}^{\infty} \varphi^{\nu}(r)\, z^{2\nu} \qquad (2:2)$$

we obtain

$$\mathfrak{E}_r(r, z) = -\frac{\partial}{\partial r} \Phi(r, z) = -\sum_{\nu=0}^{\infty} \varphi_\nu'(r)\, z^{2\nu} \qquad (2:3)$$

$$\mathfrak{E}_z(r, z) = -\frac{\partial}{\partial z} \Phi(r, z) = -\sum_{\nu=1}^{\infty} \varphi_\nu(r)\, 2\nu z^{2\nu-1} \qquad (2:4)$$

and by means of the condition div $\mathfrak{E} = 0$

$$\frac{1}{r}\frac{\partial}{\partial r} r\,\mathfrak{E}_r(r, z) + \frac{\partial}{\partial z}\mathfrak{E}_z(r, z) =$$

$$= -\sum_{\nu=0}^{\infty}\left[\frac{1}{r}\frac{\partial}{\partial r} r\,\varphi_\nu'(r) + (2\nu + 1)(2\nu + 2)\,\varphi_{\nu+1}(r)\right] z^{2\nu} = 0 \qquad (2:5)$$

whence

$$\varphi_\nu(r) = \frac{1}{2\nu!}\left(-\frac{1}{r}\frac{d}{dr}\, r\, \frac{d}{dr}\right)^{\nu} \varphi_0(r). \qquad (2:6)$$

From Eqs. 2—6 we get the following explicit formulae

$$\Phi(r, z) = -E_0 r_0 [\varrho + \tfrac{1}{2}\beta_1\varrho^2 + \tfrac{1}{3}\beta_2\varrho^3 + \tfrac{1}{4}\beta_3\varrho^4 - \tfrac{1}{2}(1 + \beta_1)\zeta^2 +$$
$$+ \tfrac{1}{2}(1 - \beta_1 - 2\beta_2)\varrho\zeta^2 - \tfrac{1}{2}(1 - \beta_1 + \beta_2 + 3\beta_3)\varrho^2\zeta^2 + \tfrac{1}{24}(1 - \beta_1 + 4\beta_2 + 6\beta_3)\zeta^4] \qquad (2:7)$$

$$\mathfrak{E}_r(r, z) = E_0 [1 + \beta_1\varrho + \beta_2\varrho^2 + \beta_3\varrho^3 +$$
$$+ \tfrac{1}{2}(1 - \beta_1 - 2\beta_2)\zeta^2 - (1 - \beta_1 + \beta_2 + 3\beta_3)\varrho\zeta^2] \qquad (2:8)$$

$$\mathfrak{E}_z(r, z) = E_0 [-(1 + \beta_1)\zeta + (1 - \beta_1 - 2\beta_2)\varrho\zeta - (1 - \beta_1 + \beta_2 + 3\beta_3)\varrho^2\zeta +$$
$$+ \tfrac{1}{6}(1 - \beta_1 + 4\beta_2 + 6\beta_3)\zeta^3] \qquad (2:9)$$

where $\zeta = z/r_0$. The potential is put equal to zero for $r = r_0$, $z = 0$.

For the purpose of realizing such a field we need also an explicit formula for the shape of the equipotential surfaces. Putting

$$\Phi(r, z) = - E_0 r_0 \sigma \tag{2:10}$$

where σ is a constant, we may calculate ϱ as a function of ζ^2. By means of successive approximations we obtain

$$\varrho = \sigma - \tfrac{1}{2}\beta_1 \sigma^2 + (\tfrac{1}{2}\beta_1^2 - \tfrac{1}{3}\beta_2)\sigma^3 + \tfrac{1}{2}[1 + \beta_1 - (1 + \beta_1^2 - 2\beta_2)\sigma]\zeta^2. \tag{2:11}$$

The curvature of a radial section of the equipotential in $\sigma = \sigma_1$, $\zeta = 0$ then is

$$\frac{1}{R_2} = 1 + \beta_1 - (1 + \beta_1^2 - 2\beta_2)\sigma_1 \tag{2:12}$$

and the mean curvature in the same point

$$\frac{1}{R_1} + \frac{1}{R_2} = \beta_1 - (\beta_1^2 - 2\beta_2)\sigma_1. \tag{2:13}$$

We may notice the special case

$$2\beta_2 = 1 + \beta_1^2 \tag{2:14}$$

where all equipotentials have the same curvature.

3. The magnetic field

An analogous calculation of the magnetic field may be started from a series expansion of the axial field component in the xy-plane

$$\mathfrak{H}_z(r, 0) = H(r) = H_0(1 + \alpha_1 \varrho + \alpha_2 \varrho^2 + \alpha_3 \varrho^3 + \cdots). \tag{3:1}$$

Even in this case the potential of the field

$$\Psi(r, z) = \sum_{\nu=0}^{\infty} \psi_\nu(r)\, z^{2\nu+1} \tag{3:2}$$

and the field components

$$\mathfrak{H}_r(r, z) = -\frac{\partial}{\partial r}\Psi(r, z) = -\sum_{\nu=0}^{\infty} \psi_\nu'(r)\, z^{2\nu+1} \tag{3:3}$$

$$\mathfrak{H}_z(r, z) = -\frac{\partial}{\partial z}\Psi(r, z) = -\sum_{\nu=0}^{\infty} \psi_\nu(r)(2\nu + 1)\, z^{2\nu} \tag{3:4}$$

are calculated for an annular region in the vicinity of $r = r_0$, $z = 0$. In this case the condition $\operatorname{div} \mathfrak{H} = 0$ gives

$$\psi_0(r) = -H(r) \tag{3:5}$$

$$\psi_\nu(r) = \frac{1}{(2\nu+1)!}\left(-\frac{1}{r}\frac{d}{dr}\,r\,\frac{d}{dr}\right)^\nu \psi_0(r) \tag{3:6}$$

and finally

$$\begin{aligned}\Psi(r,z) = -H_0 r_0 [\zeta + \alpha_1 \varrho \zeta + \alpha_2 \varrho^2 \zeta + \alpha_3 \varrho^3 \zeta - \tfrac{1}{6}(\alpha_1 + 2\alpha_2)\zeta^3 + \\ + \tfrac{1}{6}(\alpha_1 - 2\alpha_2 - 6\alpha_3)\varrho\zeta^3]\end{aligned} \tag{3:7}$$

$$\mathfrak{H}_r(r,z) = H_0[\alpha_1\zeta + 2\alpha_2\varrho\zeta + 3\alpha_3\varrho^2\zeta + \tfrac{1}{6}(\alpha_1 - 2\alpha_2 - 6\alpha_3)\zeta^3] \tag{3:8}$$

$$\mathfrak{H}_z(r,z) = H_0[1 + \alpha_1\varrho + \alpha_2\varrho^2 + \alpha_3\varrho^3 - \tfrac{1}{2}(\alpha_1 + 2\alpha_2)\zeta^2 + \tfrac{1}{2}(\alpha_1 - 2\alpha_2 - 6\alpha_3)\varrho\zeta^2]. \tag{3:9}$$

The field components can also be derived from the vector potential

$$\begin{aligned}A(r,z) = H_0 r_0 [\varrho - \tfrac{1}{2}(1-\alpha_1)\varrho^2 + \tfrac{1}{6}(3 - \alpha_1 + 2\alpha_2)\varrho^3 - \tfrac{1}{12}(6 - 2\alpha_1 + \alpha_2 - 3\alpha_3)\varrho^4 - \\ - \tfrac{1}{2}\alpha_1\zeta^2 - \alpha_2\varrho\zeta^2 - \tfrac{3}{2}\alpha_3\varrho^2\zeta^2 - \tfrac{1}{24}(\alpha_1 - 2\alpha_2 - 6\alpha_3)\zeta^4]\end{aligned} \tag{3:10}$$

by means of the relations

$$\mathfrak{H}_r = -\frac{\partial}{\partial z}A(r,z), \qquad \mathfrak{H}_z = \frac{1}{r}\frac{\partial}{\partial r}\,r\,A(r,z). \tag{3:11}$$

As in the electrical case the equations of the equipotentials are calculated from the scalar potential. We put

$$\Psi(r,z) = -H_0 r_0 \sigma \tag{3:12}$$

and compute ζ as a function of ϱ by successive approximations

$$\zeta = \sigma + \tfrac{1}{6}(\alpha_1 + 2\alpha_2)\sigma^3 - \alpha_1\sigma\varrho + (\alpha_1^2 - \alpha_2)\sigma\varrho^2. \tag{3:13}$$

This equation also determines the shape of the pole surfaces if iron of large permeability is used to produce the field.

4. The equations of motion and their solution

If a particle of mass m and electric charge e is moving in an electric field $\mathfrak{E}$ and a magnetic field $\mathfrak{H}$, the laws of motion in cylindrical coordinates are expressed (in non-relativistic approximation) by

$$m(\ddot{r} - r\dot{\varphi}^2) = -r\dot{\varphi}\,e\,\mathfrak{H}_z(r,z) + e\,\mathfrak{E}_r(r,z) \tag{4:1}$$

$$m(r\ddot{\varphi} + 2\dot{r}\dot{\varphi}) = \dot{r}\,e\,\mathfrak{H}_z(r,z) - \dot{z}\,e\,\mathfrak{H}_r(r,z) \tag{4:2}$$

$$m\ddot{z} = r\dot{\varphi}\,e\,\mathfrak{H}_r(r,z) + e\,\mathfrak{E}_z(r,z). \tag{4:3}$$

Eq. (4:2) can be integrated once with respect to the time coordinate t:

$$r^2 \dot{\varphi} = \frac{e}{m} r A(r, z) + C \tag{4:4}$$

where $A(r, z)$ is the vector potential of the magnetic field defined by Eq. (3:10). The integration constant C is determined from the initial conditions: the particle is assumed to proceed from the point $r = r_0$, $z = 0$, $\varphi = 0$ with the velocity v_1 and with the inclination γ from the tangent of the circle $r = r_0$, $z = 0$ in the same point. This gives

$$C = r_0 v_1 \cos \gamma = r_0 u_1. \tag{4:5}$$

Now $\dot{\varphi}$ is eliminated from Eqs. (1) and (3) and t is substituted by φ as an independent variable by means of Eq. (4), for example

$$\ddot{r} = \dot{\varphi} \frac{d}{d\varphi} \dot{\varphi} \frac{dr}{d\varphi} = \dot{\varphi}^2 r'' + \dot{\varphi} \frac{d\dot{\varphi}}{d\varphi} r'. \tag{4:6}$$

Thus the equations of motion are reduced to

$$r'' + B^{-1} \left(r' \frac{\partial B}{\partial r} + z' \frac{\partial B}{\partial z} \right) r' - r = - r B^{-1} \frac{e}{m} \mathfrak{H}_z + B^{-2} \frac{e}{m} \mathfrak{E}_r \tag{4:7}$$

$$z'' + B^{-1} \left(r' \frac{\partial B}{\partial r} + z' \frac{\partial B}{\partial z} \right) z' = r B^{-1} \frac{e}{m} \mathfrak{H}_r + B^{-2} \frac{e}{m} \mathfrak{E}_z \tag{4:8}$$

where

$$B = \frac{e}{m r} A + \frac{r_0 u_1}{r^2}. \tag{4:9}$$

It is seen from Eqs. (5), (7) and (8) that the circle $r = r_0$, $z = 0$ is a possible trajectory if the velocity v_1 has a value v_0 determined by

$$r_0 = r_0 B_0^{-1} \frac{e}{m} H_0 - B_0^{-2} \frac{e}{m} E_0 \tag{4:10}$$

and

$$B_0 = \frac{u_1}{r_0} = \frac{v_0}{r_0} \tag{4:11}$$

or

$$\frac{m v_0^2}{r_0} = e v_0 H_0 - e E_0. \tag{4:12}$$

The latter equation expresses the balance between the centrifugal force and the magnetic and electrostatic forces and may be considered the zeroth approximation of the solution of Eqs. (7) and (8). In order to obtain higher approximations both membra of the same equations are expanded in powers of $\varrho = (r - r_0)/r_0$ and $\zeta = z/r_0$ giving

$$\varrho'' + \lambda_r \varrho = F_0 + F_2 + F_3 + \cdots \tag{4:13}$$

$$\zeta'' + \lambda_z \zeta = G_0 + G_2 + G_3 + \cdots \tag{4:14}$$

where the first order terms in ϱ and ζ are assembled in the left membra, and all other terms in the right membra. The functions F and G may also be developed into series with respect to the velocity difference $\delta \cdot v_0 = v_1 - v_0$ and the angle γ. One should notice here that the functions are of the first order in δ and the second order in γ at least.

5. Directional focusing

a) The focusing angles

The characteristic feature of the geometrical focusing is most easily explained by first putting $v_1 = v_0$, *i. e.* assuming that a particle of velocity v_0, determined by Eq. (4 : 12), emerges from the point $r = r_0$, $z = 0$ on the central line with an inclination γ to the tangent of the central line in the same point. If terms of higher order than the first in ϱ and ζ and γ are neglected, we obtain after some calculation from Eqs. (4 : 7) and (4 : 8)

$$\varrho'' + \lambda_r \varrho = 0 \tag{5:1}$$

$$\zeta'' + \lambda_z \zeta = 0 \tag{5:2}$$

where

$$\lambda_r = 1 + (\varkappa - 1)(\varkappa - 2) + \alpha_1 \varkappa + \beta_1 (1 - \varkappa) \tag{5:3}$$

$$\lambda_z = \varkappa - 1 - \alpha_1 \varkappa - \beta_1 (1 - \varkappa) \tag{5:4}$$

and

$$\varkappa = e r_0 H_0 / m v_0 \tag{5:5}$$

is the ratio between the magnetic force and the centrifugal force.

Eqs. (1) and (2) show that ϱ and ζ are periodical functions of the angle φ if λ_r and λ_z are positive. In that case the particle will pass through the cylindrical surface $r = r_0$ after an angle $\varPhi_r^{(n)} = \pi n / \sqrt{\lambda^r}$ $(n = 1, 2, \ldots)$ and through the symmetry plane $z = 0$ after an angle $\varPhi_z^{(n)} = \pi m / \sqrt{\lambda_z}$ $(m = 1, 2, \ldots)$. It follows that the so-called focusing angles

$$\varPhi_r = \varPhi_r^{(0)} = \pi [1 + (\varkappa - 1)(\varkappa - 2) + \alpha_1 \varkappa + \beta_1 (1 - \varkappa)]^{-\frac{1}{2}} \tag{5:6}$$

$$\varPhi_z = \varPhi_z^{(0)} = \pi [\varkappa - 1 - \alpha_1 \varkappa - \beta_1 (1 - \varkappa)]^{-\frac{1}{2}} \tag{5:7}$$

only depend upon the inhomogeneities of the magnetic and electrostatic fields and the ratio $\varkappa$. The planes $\varphi = \varPhi_r$ and $\varphi = \varPhi_z$ are called focal planes for radial and axial focusing respectively. The simple relationship

$$\frac{1}{\varPhi_r^2} + \frac{1}{\varPhi_z^2} = \frac{1 + (\varkappa - 1)^2}{\pi^2} \tag{5:8}$$

shows that the focusing angles may have all values between $\pi [1 + (\varkappa - 1)^2]^{-\frac{1}{2}}$ and ∞. A smaller value for one direction is always accompanied by a de-focusing in the other direction. For double directional focusing, $\Phi_r = \Phi_z = \Phi$, the largest possible focusing angle $\Phi = \pi\sqrt{2}$ occurs at $\varkappa = 1$, that is a pure magnetic field.

b) Radial focusing

The condition for radial focusing $\Phi_z = \infty$ or $\lambda_z = 0$ gives according to Eq. (4)

$$\varkappa = (1 + \beta_1)/(1 - \alpha_1 + \beta_1) \tag{5:9}$$

or, by means of Eq. (5),

$$e\,v_0\,H_0/e\,E_0 = \varkappa/(\varkappa - 1) = (1 + \beta_1)/\alpha_1. \tag{5:10}$$

This result includes the following special cases some of which are well known

α) $\varkappa = 1$, $\alpha_1 = 0$, pure magnetic field, homogeneous and having the focusing angle $\Phi_r = \pi$.

β) $\varkappa = 0$, pure electrostatic field with $\beta_1 = -1$, *i.e.* the radial field between two cylindrical and concentric conductors. The focusing angle is $\Phi_r = \pi/\sqrt{2} \sim 127°$. This focusing principle was proposed by HUGHES and ROJANSKY (1929) and has applications within different fields especially in precision mass spectroscopy.

γ) $\varkappa = 2$, simultaneous magnetic and electrostatic fields whose gradients are coupled by the condition $\beta_1 = 2\,\alpha_1 - 1$. The focusing angle is $\Phi = \pi/\sqrt{2}$. The subcase $\alpha_1 = 0$, $\beta_1 = -1$, that is homogeneous magnetic field and radial electric field was proposed by BARTKY and DEMPSTER (1929). It has found applications in the determination of e/m for electrons by SHAW (1933, 1938) and in a mass spectrometer constructed by BONDY, JOHANNSEN and POPPER (1935). A characteristic property of all field combinations for which $\varkappa = 2$ is the first order velocity focusing, which will be discussed in Section 6.

δ) $\varkappa \to \infty$. This case corresponds to an infinitely small focusing angle or to a rectilinear central line. The special case $\alpha_1 = 0$, $\beta_1 = -1$ corresponds to the well known trochoidal motion.

c) Axial focusing

Since there seems to be no obvious applications of a pure axial focusing, we are content with an enumeration of the special cases corresponding to the cases *α*, *β*, *γ* for radial focusing.

α) $\varkappa = 1$ gives $\alpha_1 = -1$ and $\Phi_z = \pi$

β) $\varkappa = 0$ gives $\beta_1 = -3$ and $\Phi_z = \pi/\sqrt{2}$

γ) $\varkappa = 2$ gives $\beta_1 = 2\,\alpha_1 + 1$ and $\Phi_z = \pi/\sqrt{2}$.

d) Double directional focusing

The condition for equally strong focusing in the radial as well as in axial direction gives according to Eqs. (5:6) and (5:7)

$$\Phi_r = \Phi_z = \Phi = \pi \left[\frac{2}{1 + (\varkappa - 1)^2} \right]^{\frac{1}{2}} \tag{5:11}$$

and

$$\alpha_1 \varkappa + \beta_1 (1 - \varkappa) + \tfrac{1}{2} (\varkappa - 2)^2 = 0. \tag{5:12}$$

We notice the following important subcases.

α) $\varkappa = 1$, which gives $\alpha_1 = -\frac{1}{2}$ and $\Phi = \pi \sqrt{2} \sim 255°$. This case was applied in the large β-spectrometer constructed by K. Siegbahn and the present author (1946, 1949). The magnetic field strength decreases outwards in such a way that its first derivative is the same as for the function $1/\sqrt{r}$.

β) $\varkappa = 0$, which gives $\beta_1 = -2$ and $\Phi = \pi$. This case is the electrostatic counterpart to case *α*). An example of such a field is the electric field in a spherical condenser. Cf. PURCELL (1938).

γ) $\varkappa = 2$, whence $\beta_1 = 2\alpha_1$ and $\Phi = \pi$. As in the case of radial (and axial) focusing there exists an infinite number of field combinations, which also give velocity focusing of the first order. Among these cases we observe especially 1) $\alpha_1 = \beta_1 = 0$, which means a homogeneous magnetic field combined with a radially constant electric field (not a cylinder field), 2) $\alpha_1 = -\frac{1}{2}$, $\beta_1 = -1$, which is an electric cylinder field combined with a magnetic field of the type $1/\sqrt{r}$.

Since one of the quantities α_1 and β_1 may be fixed arbitrarily there is one degree of freedom left for reducing the aberrations. This possibility will be discussed in Section 8.

6. Velocity focusing

We now make the more general assumption that the initial velocity v_1, differs by $\delta \cdot v_0 = v_1 - v_0$ from the velocity v_0 determined by Eq. (4:12). Series expansion of the functions F_0 and G_0 in powers of γ now gives

$$F_0 = \delta \cdot \frac{2 - \varkappa + \delta}{(1 + \delta)^2} - \frac{(2 - \varkappa + \varkappa\delta)}{(1 + \delta)^2} \cdot \frac{\gamma^2}{2}, \quad G_0 = 0 \tag{6:1}$$

where terms of higher order than three are neglected. If only first order terms in ϱ, ζ and γ are considered, we obtain the following differential equations

$$\varrho'' + \bar{\lambda}_r \varrho = \delta \cdot \frac{2 - \varkappa + \delta}{(1 + \delta)^2} \tag{6:2}$$

$$\zeta'' + \bar{\lambda}_z \zeta = 0. \tag{6:3}$$

In general the coefficients $\bar{\lambda}_r$ and $\bar{\lambda}_z$ are functions of δ and γ, but if terms of higher order than the first in γ and δ are neglected, we have the formulae

$$\bar{\lambda}_r = \lambda_r + \delta [-8 + 11\varkappa - 4\varkappa^2 - \varkappa\alpha_1 - 2(1 - \varkappa)\beta_1] \tag{6:4}$$

$$\bar{\lambda}_z = \lambda_z + \delta [2 - 2\varkappa + \varkappa\alpha_1 + 2(1 - \varkappa)\beta_1] \tag{6:5}$$

where λ_r and λ_z are given by Eqs. (5:3) and (5:4). The complete solutions of Eqs. (6:2) and (6:3) under the above-mentioned initial conditions are

$$\varrho = \frac{\gamma}{\sqrt{\lambda_r}} \cos\theta \sin \varphi \sqrt{\lambda_r} + \frac{\delta}{\bar{\lambda}_r} \cdot \frac{2 - \varkappa + \delta}{(1+\delta)^2} (1 - \cos \varphi \sqrt{\lambda_r}) \qquad (6:6)$$

$$\zeta = \frac{\gamma}{\sqrt{\lambda_z}} \sin\theta \sin\varphi \sqrt{\lambda_z} \qquad (6:7)$$

where θ is the angle between the osculating plane of the orbit and the symmetry plane $z = 0$ in the starting point $\varphi = 0$. In the focal plane for radial focusing $\varphi = \pi/\sqrt{\lambda_r}$ ϱ has the value

$$\varrho_1 = [(2-\varkappa) + (2\varkappa - 3)\delta]\, 2\delta/\bar{\lambda}_r + $$
$$+ \frac{\pi}{2}[8 - 11\varkappa + 4\varkappa^2 + \varkappa\alpha_1 + 2(1-\varkappa)\beta_1]\gamma\delta\lambda_r^{-\frac{1}{2}} \cos\theta \qquad (6:8)$$

where terms of higher order than the second in δ are neglected. This equation shows that the focusing system has a dispersion proportional to $(2 - \varkappa)/\lambda_r$. In the important special case $\varkappa = 2$ the linear dispersion disappears and Eq. (6:8) simplifies to

$$\varrho_1 = \pi(1 + \alpha_1 - \beta_1)\gamma\delta\lambda_r^{-\frac{1}{2}} \cos\theta + 2\delta^2/\lambda_r. \qquad (6:9)$$

This result can also be expressed in the following way. If the electric force just balances the centrifugal force, then the dispersive effect of the electric field eliminates the dispersion of the magnetic field. Formula (6:9) was derived in the special case $\alpha_1 = 0$, $\beta_1 = -1$, $\lambda_r = 2$, $\theta = 0$ by BARTKY and DEMPSTER (1929).

7. The geometrical aberrations for a point object

We return to the assumption $v_1 = v_0$ and compute the functions F_2 and G_2 defined by Eqs. (4:13) and (4:14) by a further expansion of both membra of Eqs. (4:7) and (4:8) and obtain

$$F_2 = -(\varkappa - 2)\varrho'^2 + f_{11}\varrho^2 + f_{22}\zeta^2 \qquad (7:1)$$

where

$$f_{11} = (\varkappa-2)(2\varkappa^2 - \tfrac{9}{2}\varkappa + 3) + \frac{\varkappa}{2}(\varkappa-4)\alpha_1 - 2(\varkappa-1)(\varkappa-2)\beta_1 - \varkappa\alpha_2 - (1-\varkappa)\beta_2 \quad (7:2)$$

$$f_{22} = \tfrac{1}{2}(\varkappa-1) + \frac{\varkappa}{2}(\varkappa-1)\alpha_1 + \tfrac{1}{2}(1-\varkappa)\beta_1 + \varkappa\alpha_2 + (1-\varkappa)\beta_2 \qquad (7:3)$$

and

$$G_2 = -(\varkappa - 2)\varrho'\zeta' + g_{12}\varrho\zeta \qquad (7:4)$$

where

$$g_{12} = (\varkappa-1)(2\varkappa-3) + \varkappa(3-\varkappa)\alpha_1 + (\varkappa-1)(2\varkappa-5)\beta_1 + 2\varkappa\alpha_2 + 2(1-\varkappa)\beta_2. \quad (7:5)$$

The differential equations

$$\varrho'' + \lambda_r \varrho = F_0 + F_2 \tag{7:6}$$

$$\zeta'' + \lambda_z \zeta = G_0 + G_2 \tag{7:7}$$

where now

$$F_0 = \tfrac{1}{2}(\varkappa - 2)\gamma^2, \qquad G_0 = 0, \tag{7:8}$$

are solved by a method of successive approximations. In the first approximation where the right membra of Eqs. (7 : 6) and (7 : 7) are put equal to zero the solutions are

$$\varrho^{(0)} = \frac{\gamma}{\sqrt{\lambda_r}} \cos\theta \sin\varphi\sqrt{\lambda_r} \tag{7:9}$$

$$\zeta^{(0)} = \frac{\lambda}{\sqrt{\lambda_z}} \sin\theta \sin\varphi\sqrt{\lambda_z} \tag{7:10}$$

where the initial conditions

$$\varrho^{(0)} = 0,\ \zeta^{(0)} = 0,\ \varrho^{(0)\prime} = \gamma\cos\theta,\ \zeta^{(0)\prime} = \gamma\sin\theta \ \text{ for } \varphi = 0 \tag{7:11}$$

are adopted. Next approximation is obtained by substituting the unknown functions ϱ and ζ in F_2 and G_2 by the first order solutions (7 : 9) and (7 : 10). From the inhomogeneous differential equations so obtained we determine particular solutions, which satisfy the following initial conditions

$$\varrho^{(1)} = \zeta^{(1)} = \varrho^{(1)\prime} = \zeta^{(1)\prime} = 0 \quad \text{for} \quad \varphi = 0 \tag{7:12}$$

whence the complete solutions of the second order are

$$\varrho = \varrho^{(0)} + \varrho^{(1)}, \qquad \zeta = \zeta^{(0)} + \zeta^{(1)}. \tag{7:13}$$

By means of elementary calculations we find

$$\begin{aligned}\varrho^{(1)} &= \frac{1}{\lambda_r}\left[\frac{f_{11}}{\lambda_r}\cos^2\theta + \left(\varkappa - 2 + \frac{f_{22}}{\lambda_z}\right)\sin^2\theta\right]\tfrac{1}{2}\gamma^2\left(1 - \cos\varphi\sqrt{\lambda_r}\right) + \\ &+ \frac{1}{3\lambda_r}\left(\varkappa - 2 + \frac{f_{11}}{\lambda_r}\right)\tfrac{1}{2}\gamma^2\cos^2\theta\left(\cos 2\varphi\sqrt{\lambda_r} - \cos\varphi\sqrt{\lambda_r}\right) + \\ &+ \frac{1}{4\lambda_z - \lambda_r}\cdot\frac{f_{22}}{\lambda_z}\cdot\tfrac{1}{2}\gamma^2\sin^2\theta\left(\cos 2\varphi\sqrt{\lambda_z} - \cos\varphi\sqrt{\lambda_r}\right)\end{aligned} \tag{7:14}$$

$$\begin{aligned}\zeta^{(1)} = {}& \frac{1}{(\sqrt{\lambda_r} - \sqrt{\lambda_z})^2 - \lambda_z}\left(\varkappa - 2 - \frac{g_{12}}{\sqrt{\lambda_r\lambda_z}}\right)\tfrac{1}{2}\gamma^2\sin\theta\cos\theta\left[\cos\varphi\left(\sqrt{\lambda_r} - \sqrt{\lambda_z}\right) - \right. \\ & \left. - \cos\varphi\sqrt{\lambda_z}\right] + \frac{1}{(\sqrt{\lambda_r} + \sqrt{\lambda_z})^2 - \lambda_z}\left(\varkappa - 2 + \frac{g_{12}}{\sqrt{\lambda_r\lambda_z}}\right)\cdot \\ & \cdot\tfrac{1}{2}\gamma^2\sin\theta\cos\theta\left[\cos\varphi\left(\sqrt{\lambda_r} + \sqrt{\lambda_z}\right) - \cos\varphi\sqrt{\lambda_z}\right].\end{aligned} \tag{7:15}$$

It should be noticed that these formulae are applicable also in the limiting cases $\lambda_r \to 0$ and $\lambda_z \to 0$ as well as in the case of resonance $\lambda_r \to 4\lambda_z$.

Since we are especially interested in radial focusing we compute the line width in the focal plane for radial focusing by putting $\varphi = \pi/\sqrt{\lambda_r}$ in Eqs. (7:14) and (7:15)

$$\varrho_1^{(1)} = \frac{1}{\lambda_r}\left[\frac{f_{11}}{\lambda_r}\cos^2\theta + \left(\varkappa - 2 + \frac{f_{22}}{\lambda_z}\right)\sin^2\theta\right]\gamma^2 + \frac{1}{3\lambda_r}\left(\varkappa - 2 + \frac{f_{11}}{\lambda_r}\right)\gamma^2\cos^2\theta + \\ + \frac{1}{4\lambda_z - \lambda_r}\cdot\frac{f_{22}}{\lambda_z}\cdot\tfrac{1}{2}\gamma^2\sin^2\theta\left[\cos 2\pi\sqrt{\lambda_z/\lambda_r} + 1\right] \tag{7:16}$$

$$\zeta_1^{(1)} = \frac{2}{4\lambda_z - \lambda_r}\left(\varkappa - 2 - \frac{2g_{12}}{\lambda_r}\right)\gamma^2\sin\theta\cos\theta\cos\pi\sqrt{\lambda_z/\lambda_r}. \tag{7:17}$$

In the case of two-directional focusing, these expressions are simplified into

$$\varrho_1^{(1)} = \left(\frac{4}{3\lambda^2}f_{11} + \frac{\varkappa - 2}{3\lambda}\right)\gamma^2\cos^2\theta + \left(\frac{\varkappa - 2}{\lambda} + \frac{4f_{22}}{3\lambda^2}\right)\gamma^2\sin^2\theta \tag{7:18}$$

$$\zeta_1^{(1)} = -\frac{2}{3\lambda}\left(\varkappa - 2 - \frac{2g_{12}}{\lambda}\right)\gamma^2\sin\theta\cos\theta. \tag{7:19}$$

It remains to give the explicit formulae in the important subcases, which were mentioned in Section 5.

α) $\varkappa = 1,\ \lambda = \tfrac{1}{2},\ a_1 = -\tfrac{1}{2}$:

$$\varrho_1^{(1)} = -\tfrac{2}{3}(8a_2 - 1)\gamma^2\cos^2\theta + \tfrac{2}{3}(8a_2 - 3)\gamma^2\sin^2\theta. \tag{7:20}$$

$$\zeta_1^{(1)} = \tfrac{4}{3}(8a_2 - 3)\gamma^2\sin\theta\cos\theta. \tag{7:21}$$

β) $\varkappa = 0,\ \lambda = 1,\ \beta_1 = -2$:

$$\varrho_1^{(1)} = -\tfrac{2}{3}(2\beta_2 - 3)\gamma^2\cos^2\theta + \tfrac{4}{3}(\beta_2 - 3)\gamma^2\sin^2\theta. \tag{7:22}$$

$$\zeta_1^{(1)} = \tfrac{8}{3}(\beta_2 - 3)\gamma^2\sin\theta\cos\theta. \tag{7:23}$$

γ) $\varkappa = 2,\ \lambda = 1,\ \beta_1 = 2a_1$:

$$\varrho_1^{(1)} = -\tfrac{4}{3}(2a_1 + 2a_2 - \beta_2)\gamma^2\cos^2\theta + \tfrac{2}{3}(1 + 4a_2 - 2\beta_2)\gamma^2\sin^2\theta. \tag{7:24}$$

$$\zeta_1^{(1)} = \tfrac{4}{3}(1 + 4a_2 - 2\beta_2)\gamma^2\sin\theta\cos\theta. \tag{7:25}$$

8. Directional focusing of the second order

a) Two-directional focusing

Since high resolving power at a given energy or mass dispersion is a desirable feature of every focusing system, we are mainly interested in the radial line width in the focal plane for radial focusing. Or, the quality of the radial focusing is a factor determining both resolving power and transmission power of the system, while the axial focusing is important only to the transmission power. From that point of view we have to find all systems in which the radial line width $\varrho_1^{(1)}$ determined by Eq. (7 : 16), vanishes for all values of θ. According to Eqs. (7 : 2), (7 : 3) and (5 : 12) we then obtain for all values of $\varkappa$ except 0 and 1

$$\alpha_1 = \tfrac{1}{4} + (\varkappa - 2)\frac{6\varkappa^2 - 8\varkappa + 3}{4\varkappa(\varkappa - 1)} \tag{8:1}$$

$$\beta_1 = \tfrac{1}{2} + (\varkappa - 2)\cdot\frac{8\varkappa^2 - 15\varkappa + 8}{4(\varkappa - 1)^2} \tag{8:2}$$

$$\frac{\alpha_2}{1 - \varkappa} + \frac{\beta_2}{\varkappa} = \frac{\varkappa + 2}{8\varkappa} + \frac{\varkappa - 2}{8\varkappa(\varkappa - 1)}\left(9\varkappa^2 - 22\varkappa + 17 - \frac{\varkappa}{\varkappa - 1}\right)\cdot \tag{8:3}$$

These equations show that second order focusing occurs at a suitable choice of the coefficients $\alpha_1, \beta_1, \alpha_2, \beta_2$ for all values of $\varkappa$ except $\varkappa = 1$ (purely magnetic field) and $\varkappa = 0$ (purely electrostatic field). In the case of velocity focusing $\varkappa = 2$ we get

$$\alpha_1 = \tfrac{1}{4},\ \beta_1 = \tfrac{1}{2},\ \beta_2 = 2\alpha_2 + \tfrac{1}{2}. \tag{8:4}$$

From Eq. (7 : 17) it follows that the axial line width $\zeta_1^{(1)}$ also vanishes in this case. Consequently every field combination fullfilling the conditions (8 : 4) represents an almost ideal imaging system possessing both *first order velocity focusing and second order two-directional focusing.*

b) Astigmatic imaging systems

If we are not interested in axial focusing there might exist other fields, which provide only point-to-line representation but still second order radial focusing. An investigation of the case $\varkappa = 2$ shows that there are two groups of such cases, namely on one hand

$$\beta_1 = 1 - 2\alpha_1 \quad \text{and} \quad \beta_2 = 2\alpha_1 + 2\alpha_2 \tag{8:5}$$

giving

$$\lambda_r = 4\alpha_1 \quad \text{and} \quad \lambda_z = 2 - 4\alpha_1$$

and on the other hand a root of the equation

$$\cos^2 \pi q = 1 - 4q^2 \tag{8:6}$$

where $q = \sqrt{\lambda_z/\lambda_r} = \Phi_r/\Phi_z$ together with the condition

$$\beta_2 = 2\alpha_1 + 2\alpha_2. \tag{8:7}$$

In the extreme cases $\varkappa = 1$ and $\varkappa = 0$ there are no similar solutions. In a β-spectrometer constructed by Langer and Cook (1948) where the field has the coefficients $\alpha_1 = 0$, $\alpha_2 = -\frac{3}{4}$ the line width $\varrho_1^{(1)}$ is zero only for paths in the symmetry plane while all other paths $(\theta \neq 0)$ give a line width

$$\varrho_1^{(1)} = (2 - \tfrac{3}{4}\pi^2)\,\gamma^2 \sin^2 \theta. \tag{8:8}$$

This error, however, should be negligible as the axial aperture must be kept small because of the absence of axial focusing.

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